

# AN APPROXIMATION METHOD FOR THE OPTIMIZATION OF $p$ -TH MOMENT OF $\mathbb{R}^n$ -VALUED RANDOM VARIABLE

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**ABSTRACT.** This paper mainly addresses the optimization of  $p$ -th moment of  $\mathbb{R}^n$ -valued random variable. Through an ingenious approximation mechanism, one transforms the maximization problem into a sequence of minimization problems, which can be converted into a sequence of nonlinear differential equations with constraints by variational approach. The existence and uniqueness of the solution for each equation can be demonstrated by applying the canonical duality method. Moreover, the dual transformation gives a sequence of perfect dual maximization problems. In the final analysis, one constructs the approximation of the probability density function accordingly.

**RÉSUMÉ.** Dans cet article, on considère essentiellement le problème de maximisation du moment d'ordre  $p$  pour  $\mathbb{R}^n$ -vecteur aléatoire. En utilisant un mécanisme d'approximation ingénieuse, on transforme le problème en une séquence de problèmes de minimisation, qui peut être convertie en une séquence des équations différentielles nonlinéaires avec des contraintes par la méthode variationnelle. En particulier, nous prouvons l'existence et l'unicité de la solution en appliquant la méthode de dualité canonique. De plus, la transformation de dualité donne une séquence des duals parfaits de maximisation. Enfin, nous étudions l'approximation de la densité de probabilité.

## 1. INTRODUCTION

Optimization and probability theory is frequently used in modern financial studies, such as option pricing, portfolio investment, asset management etc. In practice, for instance, a portfolio manager of a mutual fund may invest in diversified securities to maximize the returns from increases in the prices of the securities on hand. Interested readers can refer to [3, 15, 16] for more details.

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In this paper, we discuss a typical abstract model in these respects, namely, maximization of the  $p$ -th moment of  $\mathbb{R}^n$ -valued random variable. Let  $\Omega = \mathbb{B}(O, R_1) \setminus \mathbb{B}(O, R_2)$ ,  $R_1 > R_2 > 0$ , where  $\mathbb{B}(O, R_1)$  and  $\mathbb{B}(O, R_2)$  denote open balls with center  $O$  and radii  $R_1$  and  $R_2$  in the Euclidean space  $\mathbb{R}^n$ , respectively. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space.  $X$  is real-valued random variable, then

$$\mathbb{E}(|X|^p) = \int_{\Omega} |X(\omega)|^p d\mathbb{P}(\omega), \quad (p > 0)$$

is said to be the  $p$ -th moment of  $X$  for  $X \in L^p(\Omega, \mathbb{R}^n)$ , which denotes the family of  $\mathbb{R}^n$ -valued random variable  $X$  with  $\mathbb{E}(|X|^p) < \infty$ . Readers can refer to [16] for more details concerned with convergence and moment inequalities in this weighted  $L^p$ -space. Let  $\alpha > 0$  be sufficiently large and consider the radially symmetric probability densities subject to the following constraints,

$$(1) \quad u \in W_0^{1,\infty}(\Omega) \cap C(\overline{\Omega}),$$

$$(2) \quad u \geq 0, \text{ a.e. in } \Omega,$$

$$(3) \quad \|u\|_{L^1(\Omega)} = 1,$$

$$(4) \quad \|\nabla u\|_{L^\infty(\Omega)} \leq \alpha,$$

where  $W_0^{1,\infty}(\Omega)$  is the Sobolev space [1]. In this paper, we focus on the following inverse problem, namely, maximization of the  $p$ -th moment of the real-valued random variable  $Y \in \Omega$  with respect to the probability densities  $u$  subject to (1)-(4),

$$(5) \quad (\mathcal{P}) : \max_u \left\{ \mathbb{E}_u(|Y|^p) := \int_{\Omega} |y|^p u(y) dy \right\}.$$

If  $p \geq 1$ , then  $|y|^p$  is a convex function on  $\Omega$ . In this sense, it is natural to apply our analysis to general convex functions, which represent typical payoff functions in the financial system. And the maximization problem (5) is aimed to find an optimal investment strategy in order to maximize the profit. Furthermore, the constraint (1)  $u = 0$  on  $\partial\Omega$  requires the manager to invest in neither the most profitable but highly risky nor unprofitable projects. While the constraint (4) establishes the principle of a diversified investment portfolio, which means, it is not allowed to put all eggs in one basket. Our discussion can also be applied in the improvement of international migration process. In this case, the sociological meaning of (5) is to maximize the social benefits by choosing an appropriate distribution density of the population.

Indeed, many mathematical tools have been developed for the infinite-dimensional linear programming, such as Monge-Kantorovich-Rubinstein-Wasserstein matrices [13, 14], etc. In this paper, we investigate the analytic approximating probability density through *canonical duality method* introduced by David Y. Gao and G. Strang [6, 7, 8]. This theory was originally proposed to find minimizers for a non-convex strain energy functional with a double-well potential. During the last few years, considerable effort has been taken to illustrate these non-convex problems from the theoretical viewpoint. Through applying this method, they characterized the local energy extrema and the global energy minimizer for both hard device and soft device and finally obtained the

analytical solutions [9, 10, 11].

Inspired by the survey paper [4], we propose an approximation approach of nonlinear differential equation by introducing a sequence of approximation problems for the primal problem  $(\mathcal{P})$ , namely,

$$(6) \quad (\mathcal{P}^{(\varepsilon)}) : \min_{w_\varepsilon} \left\{ I^{(\varepsilon)}[w_\varepsilon] := \int_{\Omega} L^{(\varepsilon)}(\nabla w_\varepsilon, w_\varepsilon, y) dy := \int_{\Omega} \left( H^{(\varepsilon)}(\nabla w_\varepsilon) - w_\varepsilon |y|^p \right) dy \right\},$$

where  $H^{(\varepsilon)} : \mathbb{R}^n \rightarrow \mathbb{R}^+$  is defined as

$$H^{(\varepsilon)}(\gamma) := \varepsilon e^{(|\gamma|^2 - \alpha^2)/(2\varepsilon)},$$

and  $w_\varepsilon$  is subject to the constraints (1)-(4). Moreover,

$$L^{(\varepsilon)}(P, z, y) : \mathbb{R}^n \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$$

satisfies the following coercivity inequality and is convex in the variable  $P$ ,

$$L^{(\varepsilon)}(P, z, y) \geq p_\varepsilon |P|^2 - q_\varepsilon, \quad P \in \mathbb{R}^n, z \in \mathbb{R}, y \in \Omega,$$

for certain constants  $p_\varepsilon$  and  $q_\varepsilon$ .  $I^{(\varepsilon)}$  is called the potential energy functional and is weakly lower semicontinuous on  $W_0^{1,\infty}(\Omega)$ . It's worth noticing that when  $|\gamma| \leq \alpha$ , then

$$\lim_{\varepsilon \rightarrow 0^+} H^{(\varepsilon)}(\gamma) = 0$$

uniformly. Consequently, once such a sequence of functions  $\{\bar{u}_\varepsilon\}_\varepsilon$  satisfying

$$I^{(\varepsilon)}[\bar{u}_\varepsilon] = \min_{w_\varepsilon} \left\{ I^{(\varepsilon)}[w_\varepsilon] \right\}$$

is obtained, then it will help find an optimal probability density which solves the primal problem  $(\mathcal{P})$ . The key mission is to obtain an explicit representation of this approximation sequence  $\{\bar{u}_\varepsilon\}_\varepsilon$ . Generally speaking, there are plenty of approximating schemes, for example, one can also let

$$H^{(\varepsilon)}(\gamma) := \varepsilon(|\gamma|^2 - \alpha^2)^2.$$

Then by following the procedure of dealing with double-well potentials in [6], one could definitely find an optimal probability density.

By variational calculus, correspondingly, one derives a sequence of Euler-Lagrange equations for  $(\mathcal{P}^{(\varepsilon)})$ ,

$$(7) \quad \operatorname{div}(e^{(|\nabla \bar{u}_\varepsilon|^2 - \alpha^2)/(2\varepsilon)} \nabla \bar{u}_\varepsilon) + |y|^p = 0, \quad \text{in } U^{(\varepsilon)},$$

equipped with the Dirichlet boundary condition, where the compact support

$$U^{(\varepsilon)} := \operatorname{Supp}(\bar{u}_\varepsilon) \subset \Omega$$

is connected and will be determined in Lemma 2.9. The term  $e^{(|\nabla u_\varepsilon|^2 - \alpha^2)/(2\varepsilon)}$  is called the transport density. Clearly, similar as  $p$ -Laplacian,  $e^{(|\nabla u_\varepsilon|^2 - \alpha^2)/(2\varepsilon)}$  is a highly nonlinear function with respect to  $\nabla u_\varepsilon$ , which is difficult to solve by the direct approach [2, 5]. However, by the canonical duality theory, one is able to demonstrate the existence and uniqueness of the solution of the Euler-Lagrange equation, which establishes the equivalence between the global minimizer of  $(\mathcal{P}^{(\varepsilon)})$  and the solution of Euler-Lagrange

equation (7).

Now we introduce the main theorem in this paper.

**Theorem 1.1.** *Let  $\{y : |y| = R_1\} \subset U^{(\varepsilon)}$  and  $R_1 \gg R_2$ . For any  $\varepsilon > 0$ , there exists a unique radially symmetric solution  $\bar{u}_\varepsilon$  satisfying the constraints (1)-(4) for the Euler-Lagrange equation (7). At the same time,  $\bar{u}_\varepsilon$  is a global minimizer for the approximation problem (6) in the following explicit form  $\bar{u}_\varepsilon(r)$  (without any confusion with respect to  $\bar{u}_\varepsilon(y)$ ),*

$$\bar{u}_\varepsilon(r) = \begin{cases} \int_{R_1}^r \rho^{1-n} (C_\varepsilon(R_1) - G(\rho)) / E_\varepsilon^{-1} (\rho^{2-2n} (C_\varepsilon(R_1) - G(\rho))^2) d\rho, & r \in [p_\varepsilon^*(R_1), R_1], \\ 0, & \text{elsewhere in } \Omega; \end{cases}$$

where  $E_\varepsilon$  and  $G$  are defined as

$$\begin{cases} E_\varepsilon(x) := x^2 \ln(e^{\alpha^2} x^{2\varepsilon}), & x \in [e^{-\alpha^2/(2\varepsilon)}, 1], \\ G(r) := r^{n+p}/(n+p), & r \in [p_\varepsilon^*(R_1), R_1]. \end{cases}$$

$E_\varepsilon^{-1}$  stands for the inverse of  $E_\varepsilon$ , both  $C_\varepsilon(R_1)$  and  $p_\varepsilon^*(R_1)$  are constants depending on the radius  $R_1$  and  $\varepsilon$ . Furthermore, by letting  $\varepsilon \rightarrow 0^+$ , one can solve the optimization problem (5) for the  $p$ -th moment of the real-valued variable  $Y \in \Omega$ . That is to say, for the maximization problem (P), there exists a global probability density maximizer which satisfies the constraints (1)-(4).

**Remark 1.2.** We require  $R_1 \gg R_2$  such that, for any  $\varepsilon > 0$ ,

$$(8) \quad \int_{R_2}^{R_1} r^{n-1} \int_{R_1}^r \rho^{1-n} (C_\varepsilon - G(\rho)) / E_\varepsilon^{-1} (\rho^{2-2n} (C_\varepsilon - G(\rho))^2) d\rho dr > \Gamma(n/2)/(2\pi^{n/2}),$$

where  $\Gamma(x)$  stands for the Gamma function,  $C_\varepsilon$  given in Lemma 2.8 depends on  $R_i$ ,  $i = 1, 2$ . This assumption is so important that it determines the existence of such a probability density which satisfies the normalized balance condition (3).

**Remark 1.3.** On the one hand,  $\{y : |y| = R_1\} \subset U^{(\varepsilon)}$  indicates in the financial market, venture capitalists prefer to invest in enterprises that are too risky for the standard capital markets or bank loans to get a significant return through an eventual exit event, such as IPO (initial public offerings) or trade sale of the companies. On the other hand,  $\{y : |y| = R_2\} \subset U^{(\varepsilon)}$  models the reluctance of a risk-averse investor to accept a bargain with higher risk rather than another bargain with a more certain, but possibly lower expected payoff.

**Remark 1.4.** For a general radially symmetric, positive and convex payoff function  $g(|Y|)$ , by a similar approach as in the proof of Theorem 1.1, one is able to solve the following optimization problem

$$(9) \quad \max_u \left\{ \mathbb{E}_u(g(|Y|)) := \int_{\Omega} g(|y|) u(y) dy \right\},$$

where  $u$  is subject to the constraints (1)-(4).

**Remark 1.5.** *For the pricing of options, volatility is a measure of the rate and magnitude of the change of prices (up or down) of the underlying. If volatility is high, the premium on the option will be relatively high, and vice versa. This is in relation to the following maximization of variance of  $\mathbb{R}$ -valued random variable,*

$$\max_u \left\{ \mathbb{E}_u(Y - \mathbb{E}_u(Y))^2 = \mathbb{E}_u(Y^2) - (\mathbb{E}_u(Y))^2 \right\}.$$

*If we require*

$$(10) \quad \mathbb{E}_u(Y) = \mu,$$

*then the problem is reduced to the maximization of second moment of  $\mathbb{R}$ -valued random variable with probability densities subject to (1)-(4) and (10). Following the proof of Theorem 1.1, one is able to find a probability density maximizer. If  $\mathbb{E}_u(Y)$  keeps unknown, this nonlinear optimization problem remains to be discussed theoretically.*

The rest of the paper is organized as follows. In Section 2, first, we introduce some useful notations which will simplify the proof considerably. Then, we apply the canonical dual transformation to deduce a sequence of perfect dual problems  $(\mathcal{P}_d^{(\varepsilon)})$  corresponding to  $(\mathcal{P}^{(\varepsilon)})$  and a pure complementary energy principle. Next, we apply the canonical duality theory to prove Theorem 1.1. A few remarks will conclude the discussion.

## 2. PROOF OF THEOREM 1.1: CANONICAL DUALITY APPROACH

**2.1. Useful notations.** Before proving the main result, first and foremost, we introduce some useful notations.

- $\theta_\varepsilon$  is the corresponding Gâteaux derivative of  $H^{(\varepsilon)}$  with respect to  $\nabla u_\varepsilon$ , given by

$$\theta_\varepsilon(y) = e^{(|\nabla u_\varepsilon|^2 - \alpha^2)/(2\varepsilon)} \nabla u_\varepsilon.$$

- $\Phi^{(\varepsilon)}$  is a nonlinear geometric mapping given by

$$\Phi^{(\varepsilon)}(u_\varepsilon) := (|\nabla u_\varepsilon|^2 - \alpha^2)/(2\varepsilon).$$

For convenience's sake, denote

$$\xi_\varepsilon := \Phi^{(\varepsilon)}(u_\varepsilon).$$

It is evident that  $\xi_\varepsilon$  belongs to the function space  $\mathcal{U}$  given by

$$\mathcal{U} := \left\{ \phi \mid \phi \leq 0 \right\}.$$

- $\Psi^{(\varepsilon)}$  is a canonical energy defined as

$$\Psi^{(\varepsilon)}(\xi_\varepsilon) := \varepsilon e^{\xi_\varepsilon},$$

which is a convex function with respect to  $\xi_\varepsilon$ .

- $\zeta_\varepsilon$  is the corresponding Gâteaux derivative of  $\Psi^{(\varepsilon)}$  with respect to  $\xi_\varepsilon$ , given by

$$\zeta_\varepsilon = \varepsilon e^{\xi_\varepsilon},$$

which is invertible with respect to  $\xi_\varepsilon$  and belongs to the function space  $\mathcal{V}^{(\varepsilon)}$ ,

$$\mathcal{V}^{(\varepsilon)} := \left\{ \phi \mid 0 < \phi \leq \varepsilon \right\}.$$

- $\Psi_*^{(\varepsilon)}$  is defined as

$$\Psi_*^{(\varepsilon)}(\zeta_\varepsilon) := \xi_\varepsilon \zeta_\varepsilon - \Psi^{(\varepsilon)}(\xi_\varepsilon) = \zeta_\varepsilon (\ln(\zeta_\varepsilon/\varepsilon) - 1).$$

- $\lambda_\varepsilon$  is defined as

$$\lambda_\varepsilon := \zeta_\varepsilon/\varepsilon,$$

and belongs to the function space  $\mathcal{V}$ ,

$$\mathcal{V} := \left\{ \phi \mid 0 < \phi \leq 1 \right\}.$$

## 2.2. Canonical duality techniques.

**Definition 2.1.** By Legendre transformation, one defines a Gao-Strang total complementary energy functional  $\Xi^{(\varepsilon)}$ ,

$$\Xi^{(\varepsilon)}(u_\varepsilon, \zeta_\varepsilon) := \int_{U^{(\varepsilon)}} \left\{ \Phi^{(\varepsilon)}(u_\varepsilon) \zeta_\varepsilon - \Psi_*^{(\varepsilon)}(\zeta_\varepsilon) - |y|^p u_\varepsilon \right\} dy.$$

Next, we introduce an important *criticality criterium* for  $\Xi^{(\varepsilon)}$ .

**Definition 2.2.**  $(\bar{u}_\varepsilon, \bar{\zeta}_\varepsilon)$  is called a critical pair of  $\Xi^{(\varepsilon)}$  if and only if

$$(11) \quad D_{u_\varepsilon} \Xi^{(\varepsilon)}(\bar{u}_\varepsilon, \bar{\zeta}_\varepsilon) = 0,$$

and

$$(12) \quad D_{\zeta_\varepsilon} \Xi^{(\varepsilon)}(\bar{u}_\varepsilon, \bar{\zeta}_\varepsilon) = 0,$$

where  $D_{u_\varepsilon}, D_{\zeta_\varepsilon}$  denote the partial Gâteaux derivatives of  $\Xi^{(\varepsilon)}$ , respectively.

Indeed, by variational calculus, we have the following observation from (11) and (12).

**Lemma 2.3.** On the one hand, for any fixed  $\zeta_\varepsilon \in \mathcal{V}^{(\varepsilon)}$ , (11) is equivalent to the equilibrium equation

$$\operatorname{div}(\lambda_\varepsilon \nabla \bar{u}_\varepsilon) + |y|^p = 0, \quad \text{in } U^{(\varepsilon)}.$$

On the other hand, for any fixed  $u_\varepsilon$  satisfying (1)-(4), (12) is consistent with the constructive law

$$\Phi^{(\varepsilon)}(u_\varepsilon) = D_{\zeta_\varepsilon} \Psi_*^{(\varepsilon)}(\bar{\zeta}_\varepsilon).$$

Lemma 2.3 indicates that  $\bar{u}_\varepsilon$  from the critical pair  $(\bar{u}_\varepsilon, \bar{\zeta}_\varepsilon)$  solves the Euler-Lagrange equation (7).

**Definition 2.4.** From Definition 2.1, one defines the Gao-Strang pure complementary energy  $I_d^{(\varepsilon)}$  in the form

$$I_d^{(\varepsilon)}[\zeta_\varepsilon] := \Xi^{(\varepsilon)}(\bar{u}_\varepsilon, \zeta_\varepsilon),$$

where  $\bar{u}_\varepsilon$  solves the Euler-Lagrange equation (7).

As a matter of fact, another representation of the pure energy  $I_d^{(\varepsilon)}$ , given by the following lemma, is much more useful for our purpose.

**Lemma 2.5.** *The pure complementary energy functional  $I_d^{(\varepsilon)}$  can be rewritten as*

$$I_d^{(\varepsilon)}[\zeta_\varepsilon] = -1/2 \int_{U^{(\varepsilon)}} \left\{ \varepsilon |\theta_\varepsilon|^2 / \zeta_\varepsilon + \alpha^2 \zeta_\varepsilon / \varepsilon + 2\zeta_\varepsilon (\ln(\zeta_\varepsilon / \varepsilon) - 1) \right\} dy,$$

where  $\theta_\varepsilon$  satisfies

$$(13) \quad \operatorname{div} \theta_\varepsilon + |y|^p = 0 \text{ in } U^{(\varepsilon)},$$

equipped with a hidden boundary condition.

*Proof.* Through integrating by parts, one has

$$\begin{aligned} I_d^{(\varepsilon)}[\zeta_\varepsilon] &= - \underbrace{\int_{U^{(\varepsilon)}} \left\{ \operatorname{div}(\zeta_\varepsilon \nabla \bar{u}_\varepsilon / \varepsilon) + |y|^p \right\} \bar{u}_\varepsilon dy}_{(I)} \\ &\quad - \underbrace{1/2 \int_{U^{(\varepsilon)}} \left\{ \zeta_\varepsilon |\nabla \bar{u}_\varepsilon|^2 / \varepsilon + \alpha^2 \zeta_\varepsilon / \varepsilon + 2\zeta_\varepsilon (\ln(\zeta_\varepsilon / \varepsilon) - 1) \right\} dy}_{(II)}. \end{aligned}$$

Since  $\bar{u}_\varepsilon$  solves the Euler-Lagrange equation (7), then the first part (I) disappears. Keeping in mind the definition of  $\theta_\varepsilon$  and  $\zeta_\varepsilon$ , one reaches the conclusion.  $\square$

With the above discussion, next, we establish a sequence of dual variational problems corresponding to the approximation problems  $(\mathcal{P}^{(\varepsilon)})$ .

$$(14) \quad (\mathcal{P}_d^{(\varepsilon)}) : \max_{\zeta_\varepsilon \in \mathcal{V}^{(\varepsilon)}} \left\{ I_d^{(\varepsilon)}[\zeta_\varepsilon] = -1/2 \int_{U^{(\varepsilon)}} \left\{ \varepsilon |\theta_\varepsilon|^2 / \zeta_\varepsilon + \alpha^2 \zeta_\varepsilon / \varepsilon + 2\zeta_\varepsilon (\ln(\zeta_\varepsilon / \varepsilon) - 1) \right\} dy \right\}.$$

Indeed, by calculating the Gâteaux derivative of  $I_d^{(\varepsilon)}$  with respect to  $\zeta_\varepsilon$ , one has

**Lemma 2.6.** *The variation of  $I_d^{(\varepsilon)}$  with respect to  $\zeta_\varepsilon$  leads to the dual algebraic equation (DAE), namely,*

$$(15) \quad |\theta_\varepsilon|^2 = \bar{\zeta}_\varepsilon^2 (2 \ln(\bar{\zeta}_\varepsilon / \varepsilon) + \alpha^2 / \varepsilon) / \varepsilon,$$

where  $\bar{\zeta}_\varepsilon$  is from the critical pair  $(\bar{u}_\varepsilon, \bar{\zeta}_\varepsilon)$ .

Taking into account the notation of  $\lambda_\varepsilon$ , the identity (15) can be rewritten as

$$(16) \quad |\theta_\varepsilon|^2 = E_\varepsilon(\lambda_\varepsilon) = \lambda_\varepsilon^2 \ln(e^{\alpha^2} \lambda_\varepsilon^{2\varepsilon}).$$

It is evident  $E_\varepsilon$  is monotonously increasing with respect to  $\lambda_\varepsilon \in [e^{-\alpha^2/(2\varepsilon)}, 1]$ . In effect,  $|\theta_\varepsilon|^2$  has the following asymptotic expansion by using Taylor's expansion formula for  $\ln \lambda_\varepsilon$  at the point  $\lambda_\varepsilon = 1$ .

**Lemma 2.7.** *If  $\varepsilon > 0$  is sufficiently small, then  $|\theta_\varepsilon|^2$  has the asymptotic expansion in the form of*

$$|\theta_\varepsilon|^2 = (\alpha^2 - 2\varepsilon) \lambda_\varepsilon^2 + 2\varepsilon \lambda_\varepsilon^3 + R_\varepsilon(\lambda_\varepsilon),$$

where the remainder term

$$|R_\varepsilon(\lambda_\varepsilon)| \leq \varepsilon$$

uniformly for any  $\lambda_\varepsilon \in [e^{-\alpha^2/(2\varepsilon)}, 1]$ .

**2.3. Proof of Theorem 1.1.** From the above discussion, one deduces that, once  $\theta_\varepsilon$  is given, then the analytic radially symmetric solution of the Euler-Lagrange equation (7) can be represented as

$$(17) \quad \bar{u}_\varepsilon(y) = \int_{y_0}^y \eta_\varepsilon(t) dt,$$

where  $y \in U^{(\varepsilon)}$ ,  $y_0 \in \partial U^{(\varepsilon)}$ ,  $\eta_\varepsilon = (\eta_\varepsilon^{(1)}, \eta_\varepsilon^{(2)}, \dots, \eta_\varepsilon^{(n)}) := \theta_\varepsilon / \lambda_\varepsilon$ , which satisfies the condition for path-independent integrals, namely,

$$\partial_{x_i} \eta_\varepsilon^{(j)} - \partial_{x_j} \eta_\varepsilon^{(i)} = 0, \quad i, j = 1, \dots, n.$$

In the following, one is to determine the connected compact support  $U^{(\varepsilon)}$ . Now we prove some useful lemmas as prerequisites.

**Lemma 2.8.** *For any  $\varepsilon > 0$  and any  $R \in (R_2, R_1)$ , there exists a unique smooth, radially symmetric solution  $\bar{u}_\varepsilon$  of the Euler-Lagrange equation (7) with Dirichlet boundary in the form of (17) in  $\mathbb{B}(O, R_1) \setminus \mathbb{B}(O, R)$ .*

*Proof.* Actually, in  $\mathbb{B}(O, R_1) \setminus \mathbb{B}(O, R)$ , a radially symmetric solution for the Euler-Lagrange equation (13) is of the form

$$\theta_\varepsilon = F_\varepsilon(r)(y_1, \dots, y_n) = F_\varepsilon\left(\sqrt{\sum_{i=1}^n y_i^2}\right)(y_1, \dots, y_n),$$

where

$$F_\varepsilon(r) = C_\varepsilon / r^n - r^p / (p + n)$$

is a general solution of the nonhomogeneous linear differential equation

$$rF'_\varepsilon(r) + nF_\varepsilon(r) = -r^p, \quad r \in (R, R_1),$$

where  $C_\varepsilon$  is to be determined later. From the identity (16), one sees that there exists a unique  $C^\infty$  function  $\lambda_\varepsilon \in [e^{-\alpha^2/(2\varepsilon)}, 1]$  once  $C_\varepsilon$  is given. By paying attention to the Dirichlet boundary condition, one has the radially symmetric solution  $\bar{u}_\varepsilon$  in the following form,

$$\bar{u}_\varepsilon(r) = \int_{R_1}^r \left( C_\varepsilon - G(\rho) \right) / \left( \rho^{n-1} \lambda_\varepsilon(\rho, C_\varepsilon) \right) d\rho, \quad r \in [R, R_1].$$

Recall that

$$\bar{u}_\varepsilon(R) = \int_{R_1}^{G^{-1}(C_\varepsilon)} \eta_\varepsilon(\rho, C_\varepsilon) d\rho + \int_{G^{-1}(C_\varepsilon)}^R \eta_\varepsilon(\rho, C_\varepsilon) d\rho = 0,$$

and one can determine the constant  $C_\varepsilon \in (G(R), G(R_1))$  uniquely. Indeed, let

$$\mu_\varepsilon(\rho, s) := (s - G(\rho)) / (\rho^{n-1} \lambda_\varepsilon(\rho, s))$$



and

$$M_\varepsilon(s) := \int_{R_1}^{G^{-1}(s)} \mu_\varepsilon(\rho, s) d\rho + \int_{G^{-1}(s)}^R \mu_\varepsilon(\rho, s) d\rho,$$

where  $\lambda_\varepsilon(\rho, s)$  is from (16). As a matter of fact,  $M_\varepsilon$  is strictly increasing with respect to  $s \in (G(R), G(R_1))$ , which leads to

$$C_\varepsilon = M_\varepsilon^{-1}(0).$$

As a result,  $C_\varepsilon$  depends on  $R$  and  $R_1$ . In addition, the contradiction method shows that  $C_\varepsilon$  is strictly increasing with respect to  $R \in (R_2, R_1)$ .  $\square$

With the above lemma, one is able to determine the connected compact support  $U^{(\varepsilon)}$ .

**Lemma 2.9.** *Let  $\{y : |y| = R_1\} \subset U^{(\varepsilon)}$  and  $R_1 \gg R_2$ . For any  $\varepsilon > 0$ , there exists a unique  $p_\varepsilon^*(R_1)$  such that*

$$U^{(\varepsilon)} = \overline{\mathbb{B}(O, R_1) \setminus \mathbb{B}(O, p_\varepsilon^*(R_1))}$$

and  $\bar{u}_\varepsilon$  satisfies the normalized balance condition (3).

*Proof.* Let  $\text{Supp}(\bar{u}_\varepsilon) = [s, R_1]$  and define a function

$$\Pi : (R_2, R_1) \rightarrow \mathbb{R}^+$$

as follows,

$$\Pi(s) := 2\pi^{n/2}/\Gamma(n/2) \int_s^{R_1} r^{n-1} \int_{R_1}^r \left( C_\varepsilon(s) - G(\rho) \right) / \left( \rho^{n-1} \lambda_\varepsilon(\rho, C_\varepsilon(s)) \right) d\rho dr.$$

Indeed, since  $C_\varepsilon$  is strictly increasing with respect to  $s \in (R_2, R_1)$ , consequently, it is easy to check that  $\Pi$  is a strictly decreasing function with respect to  $s \in (R_2, R_1)$ . The conclusion follows immediately when we keep in mind the assumption (8).  $\square$

In the following, we verify that  $\bar{u}_\varepsilon$  is exactly a global minimizer for  $(\mathcal{P}^{(\varepsilon)})$  and  $\bar{\zeta}_\varepsilon$  is a global maximizer for  $(\mathcal{P}_d^{(\varepsilon)})$ .

**Lemma 2.10.** *(Canonical Duality Theory) Let  $\{y : |y| = R_1\} \subset U^{(\varepsilon)}$  and  $R_1 \gg R_2$ , where  $U^{(\varepsilon)}$  is determined in Lemma 2.9. For any  $\varepsilon > 0$ ,  $\bar{u}_\varepsilon$  from Lemma 2.8 is a global minimizer for the approximation problem  $(\mathcal{P}^{(\varepsilon)})$ . And the corresponding  $\bar{\zeta}_\varepsilon$  is a global maximizer for the dual problem  $(\mathcal{P}_d^{(\varepsilon)})$ . Moreover, the following duality identity holds,*

$$(18) \quad I^{(\varepsilon)}[\bar{u}_\varepsilon] = \min_{u_\varepsilon} I^{(\varepsilon)}[u_\varepsilon] = \Xi^{(\varepsilon)}(\bar{u}_\varepsilon, \bar{\zeta}_\varepsilon) = \max_{\zeta_\varepsilon} I_d^{(\varepsilon)}[\zeta_\varepsilon] = I_d^{(\varepsilon)}[\bar{\zeta}_\varepsilon],$$

where  $u_\varepsilon$  is subject to the constraints (1)-(4) and  $\zeta_\varepsilon \in \mathcal{V}^{(\varepsilon)}$ .

Lemma 2.10 demonstrates that the maximization of the pure complementary energy functional  $I_d^{(\varepsilon)}$  is perfectly dual to the minimization of the potential energy functional  $I^{(\varepsilon)}$ . In effect, the identity (18) indicates there is no duality gap between them.

*Proof.* On the one hand, for any function  $\phi \in W_0^{1,\infty}(U^{(\varepsilon)})$ , the second variational form  $\delta_\phi^2 I^{(\varepsilon)}$  is equal to

$$(19) \quad \int_{U^{(\varepsilon)}} e^{(|\nabla \bar{u}_\varepsilon|^2 - \alpha^2)/(2\varepsilon)} \left\{ (\nabla \bar{u}_\varepsilon \cdot \nabla \phi)^2 / \varepsilon + |\nabla \phi|^2 \right\} dx.$$

On the other hand, for any function  $\psi \in \mathcal{V}^{(\varepsilon)}$ , the second variational form  $\delta_\psi^2 I_d^{(\varepsilon)}$  is equal to

$$(20) \quad - \int_{U^{(\varepsilon)}} \left\{ \varepsilon |\theta_\varepsilon|^2 \psi^2 / \bar{\zeta}_\varepsilon^3 + \psi^2 / \bar{\zeta}_\varepsilon \right\} dx.$$

From (19) and (20), one deduces immediately that

$$\delta_\phi^2 I^{(\varepsilon)}[\bar{u}_\varepsilon] \geq 0, \quad \delta_\psi^2 I_d^{(\varepsilon)}[\bar{\zeta}_\varepsilon] \leq 0.$$

□

In the final analysis, we discuss the convergence of the sequence  $\{\bar{u}_\varepsilon\}_\varepsilon$  when  $\varepsilon \rightarrow 0^+$ . According to Rellich-Kondrachov Compactness Theorem, since

$$\sup_\varepsilon |\bar{u}_\varepsilon| \leq \alpha R_1$$

and

$$\sup_\varepsilon |\nabla \bar{u}_\varepsilon| \leq \alpha,$$

then, there exists a subsequence  $\{\bar{u}_{\varepsilon_k}\}_{\varepsilon_k}$  and  $f \in W_0^{1,\infty}(\Omega) \cap C(\bar{\Omega})$  such that

$$(21) \quad \bar{u}_{\varepsilon_k} \rightarrow f \quad (k \rightarrow \infty) \text{ in } L^\infty(\Omega),$$

$$(22) \quad \nabla \bar{u}_{\varepsilon_k} \rightharpoonup \nabla f \quad (k \rightarrow \infty) \text{ weakly } * \text{ in } L^\infty(\Omega).$$

It remains to check that  $f$  satisfies (1)-(4). From (21), one knows

$$(23) \quad \bar{u}_{\varepsilon_k} \rightarrow f \quad (k \rightarrow \infty) \text{ a.e. in } \Omega.$$

According to Lebesgue's dominated convergence theorem,

$$\int_\Omega f(y) dy = \lim_{k \rightarrow \infty} \int_\Omega \bar{u}_{\varepsilon_k}(y) dy = 1.$$

From (22), one has

$$\|\nabla f\|_{L^\infty(\Omega)} \leq \liminf_{k \rightarrow \infty} \|\nabla \bar{u}_{\varepsilon_k}\|_{L^\infty(\Omega)} \leq \sup_{k \rightarrow \infty} \|\nabla \bar{u}_{\varepsilon_k}\|_{L^\infty(\Omega)} \leq \alpha.$$

Consequently, one reaches the conclusion of Theorem 1.1 by summarizing the above discussion.

**Remark 2.11.** *In this paper, we mainly focus on the construction of maximizers through the approximation procedure. Rather than the infinite-dimensional linear programming, we provide another viewpoint and give the explicit representation of the approximating probability densities by applying the canonical duality method. Furthermore, the canonical duality method proves to be useful and can also be applied in the discussion of solutions for the  $p$ -Laplacian problems etc.[12]*

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## REFERENCES

- [1] R. Adams, Sobolev Spaces, Academic Press, New York, 1975.
- [2] Bourgain J., Brezis, H.: Sur l'équation  $\operatorname{div} u = f$ . C. R. Acad. Sci. Paris, Ser. I **334**(2002), 973-976.
- [3] T. E. Copeland, J. F. Weston and K. Shastri, Financial Theory and Corporate Policy, 4th International Edition, Pearson Higher Education, 2003.
- [4] L. C. Evans, Three singular variational problems. preprint, 2002.
- [5] L. C. Evans, Partial Differential Equations. Graduate Studies in Mathematics, Vol. **19**, 2002.
- [6] D. Y. Gao, G. Strang, Geometric nonlinearity: Potential energy, complementary energy, and the gap function. Quart. Appl. Math. **47**(3)(1989), 487-504.
- [7] D. Y. Gao, Duality, triality and complementary extremum principles in non-convex parametric variational problems with applications. IMA Journal of Applied Mathematics **61**(1998), 199-235.
- [8] D. Y. Gao, Duality Principles in Nonconvex Systems: Theory, Methods and Applications. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.
- [9] D. Y. Gao, Analytic solution and triality theory for nonconvex and nonsmooth variational problems with applications. Nonlinear Analysis, **42**(7)(2000), 1161-1193.
- [10] D. Y. Gao, R. W. Ogden, G. Stravroulakis, Nonsmooth and Nonconvex Mechanics: Modelling, Analysis and Numerical Methods. Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.
- [11] D. Y. Gao, R. W. Ogden, Multiple solutions to non-convex variational problems with implications for phase transitions and numerical computation. Q. Jl Mech. Appl. Math. **61**(4)(2008), 497-522.
- [12] X. Lu and D. Y. Gao, Explicit representation for  $p$ -Laplacian problem. preprint, 2015.
- [13] S. T. Rachev, The Monge-Kantorovich mass transference problem and its stochastic applications. Theory of Prob. and Appl., **29**(1984), 647-676.
- [14] A. M. Vershik, Some remarks on the infinite-dimensional problems of linear programming, Russian Math. Survey, **25**(1970), 117-124.
- [15] P. Wilmott, Paul Wilmott On Quantitative Finance, Vol. 1-3, 2nd Edition, John Wiley & Sons Ltd., England, 2006.
- [16] S. Zhou, Nonlinear Stochastic Differential Equations, HUST Press, Wuhan, 2014.